# Comment on "Exact solutions of the derivative nonlinear Schrödinger equation for a nonlinear transmission line"

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In a recent article Kengne and Liu [Phys. Rev. E **73**, 026603 (2006)] have presented a number of exact elliptic solutions for a derivative nonlinear Schrödinger equation. It is the aim of this Comment to point out that all these solutions given in Secs. II and III of this article (referred to as KL in the following) are subcases of the general solution of Eq. (KL.9). Conditions for the parameters $A$-$E$ of the solutions given by Kengne and Liu can be found from general conditions for solitary and periodic elliptic solutions as shown in the following. Positive and bounded solutions can be found by considering the phase diagram. Therefore, the comment of Kengne and Liu that "we find its particular positive bounded solutions" can be specified.

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Equation (9) in Ref. [1] can be rewritten as

$$\left(\frac{d\zeta(z)}{dz}\right)^2 = \alpha\zeta^4 + 4\beta\zeta^3 + 6\gamma\zeta^2 + 4\delta\zeta + \epsilon \equiv R(\zeta), \tag{1}$$

with $\alpha=E$, $\beta=D/4$, $\gamma=C/6$, $\delta=K_2/4$, and $\epsilon=-4K_1^2/P^2$. As is (well) known the general solution of Eq. (1) reads [2–4]

$$\zeta(z) = \zeta_0 + \frac{\sqrt{R(\zeta_0)}\frac{d\wp(z;g_2,g_3)}{dz} + \frac{1}{2}R'(\zeta_0)\left[\wp(z;g_2,g_3) - \frac{1}{24}R''(\zeta_0)\right] + \frac{1}{24}R(\zeta_0)R'''(\zeta_0)}{2\left[\wp(z;g_2,g_3) - \frac{1}{24}R''(\zeta_0)\right]^2 - \frac{1}{48}R(\zeta_0)R''''(\zeta_0)}, \tag{2}$$

where the primes denote differentiation with respect to $\zeta$ and $\zeta_0$ is any constant, not necessarily a real root of $R(\zeta)$. If there exists a simple root $\zeta_0$ of $R(\zeta)$, Eq. (2) can be simplified to [2,5]

$$\zeta(z) = \zeta_0 + \frac{R'(\zeta_0)}{4\left[\wp(z;g_2,g_3) - \frac{1}{24}R''(\zeta_0)\right]}. \tag{3}$$

The invariants $g_2,g_3$ of Weierstrass' elliptic function $\wp(z;g_2,g_3)$ are related to the coefficients of $R(f)$ by [6]

$$g_2 = \alpha\epsilon - 4\beta\delta + 3\gamma^2, \tag{4}$$

$$g_3 = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \gamma^3 - \epsilon\beta^2. \tag{5}$$

The discriminant (of $\wp$ and $R$ [6])

$$\Delta = g_2^3 - 27g_3^2, \tag{6}$$

is suitable to classify the behavior of $\zeta(z)$. The conditions

$$\Delta \neq 0 \quad \text{or} \quad \Delta = 0, \quad g_2 > 0, \quad g_3 > 0 \tag{7}$$

lead to periodic solutions [7], whereas the conditions [7,8]

$$\Delta = 0, \quad g_2 \geqslant 0, \quad g_3 \leqslant 0 \tag{8}$$

are associated with solitary wave like solutions. Physical solutions $\zeta(z)$ must be real and bounded. Considering the phase diagram $\{\zeta, R(\zeta)\}$ [2] one obtains conditions, expressed in terms of the coefficients of Eq. (1), that determine physical solutions. Because $\zeta=a^2$ and $a$ is supposed to be real [1, p. 1] one has the additional condition that $\zeta \geqslant 0$. Therefore, it can be decided whether a solution $a$ is real and bounded. These conditions have been denoted as PDC ("phase diagram conditions") [2].

If $\Delta=0$, $\wp(z;g_2,g_3)$ can be expressed by hyperbolic, trigonometric, rational functions, respectively [8]. Thus Eq. (3) reads

$$\zeta(z) = \zeta_0 + \frac{R'(\zeta_0)}{4\left[e_1 - \frac{R''(\zeta_0)}{24} + 3e_1\,\mathrm{csch}^2(\sqrt{3e_1}z)\right]}, \quad g_3 < 0, \tag{9a}$$

$$\zeta(z) = \zeta_0 + \frac{R'(\zeta_0)}{4\left[-e_1 - \frac{R''(\zeta_0)}{24} + 3e_1\csc^2\left(\sqrt{\frac{3}{2}e_1}z\right)\right]}, \quad g_3 > 0, \tag{9b}$$

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$$\zeta(z) = \zeta_0 + \frac{6R'(\zeta_0)z^2}{24 - R''(\zeta_0)z^2}, \quad g_2 = g_3 = 0, \quad R''(\zeta_0) < 0, \tag{9c}$$

where $e_1 = (1/2)\sqrt[3]{-g_3}$ in Eq. (9a), $e_1 = (1/2)\sqrt[3]{g_3}$ in Eq. (9b), and $\zeta_0$ is a simple root of $R(\zeta)$.

Examples A and B in Sec. KL.II are subcases of Eq. (9a). The conditions for the parameters of Eq. (KL.9) follow from conditions (8) for real roots of $R(\zeta)$ due to the PDC. Furthermore the elliptic solutions in Sec. KL.III can be obtained from Eqs. (3) and (9a) [solutions resulting from Eqs. (9b) and (9c) are not contained in KL.III]. They are—of course—special cases of the general solution (2) or (3) that can be expressed in terms of the Jacobian elliptic functions instead of the Weierstrass' function. If $\Delta > 0$ Weierstrass' function $\wp$ can be expressed as [8]

$$\wp(z) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m)}, \tag{10}$$

where $m = (e_2 - e_3)/(e_1 - e_3)$ and $e_1 \geqslant e_2 \geqslant e_3$ are the roots of the equation

$$4s^3 - g_2 s - g_3 = 0. \tag{11}$$

If $\Delta < 0$ Weierstrass' function $\wp$ can be expressed as [8]

$$\wp(z) = e_2 + H_2 \frac{1 + \mathrm{cn}(2z\sqrt{H_2}, m)}{1 - \mathrm{cn}(2z\sqrt{H_2}, m)}, \tag{12}$$

where $m = 1/2 - 3e_2/4H_2$ and $H_2^2 = 3e_2^2 - g_2/4$. For instance, substitution of Eq. (10) into Eq. (3) yields [5]

$$\zeta(z) = \frac{(\alpha\zeta_0{}^3 + 4\beta\zeta_0{}^2 + 2e_3\zeta_0 + 5\gamma\zeta_0 + 2\delta)\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m) + 2(e_1 - e_3)\zeta_0}{(-\alpha\zeta_0{}^2 - 2\beta\zeta_0 + 2e_3 - \gamma)\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m) + 2(e_1 - e_3)}. \tag{13}$$

Choosing the simple root $\zeta_0$ such that

$$-\alpha\zeta_0{}^2 - 2\beta\zeta_0 + 2e_3 - \gamma = 0 \tag{14}$$

and using well known relations between the squares of Jacobian elliptic functions (cf. [9]) yields solutions of types given in Eqs. (KL.15), (KL.16), and (KL.22). Explicit solutions resulting from Eqs. (13) and (14) have been deduced in [5], Eq. (15).

Additional solutions not contained in [1] can be obtained straightforwardly by using the above method. For example, if $\alpha = \epsilon = 0$ ($E = K_1 = 0$ in [1]), the roots of Eq. (1) read

$$\zeta_1 = 0, \quad \zeta_{2,3} = \frac{-3\gamma \mp \sqrt{9\gamma^2 - 16\beta\delta}}{4\beta}. \tag{15}$$

If $\beta\delta > 0$, $\gamma < 0$ the roots $\zeta_1$ and $\zeta_2$ are suitable according to PDC [2] and

$$e_1 = \frac{1}{4}(-\gamma + \sqrt{9\gamma^2 - 16\beta\delta}),$$

$$e_2 = \frac{1}{4}(-\gamma - \sqrt{9\gamma^2 - 16\beta\delta}),$$

$$e_3 = \frac{\gamma}{2} \tag{16}$$

are the roots of Eq. (11). Inserting $\zeta_0 = \zeta_2$ into Eq. (13) and using the well known relations [9]

$$\mathrm{dn}^2(\mu z, m) + m\,\mathrm{sn}^2(\mu z, m) = 1, \tag{17}$$

$$\mathrm{sn}^2(\mu z, m) + \mathrm{cn}^2(\mu z, m) = 1, \tag{18}$$

$$\mathrm{dn}^2(\mu z, m) - m\,\mathrm{cn}^2(\mu z, m) = 1 - m \tag{19}$$

yields the periodic solution

$$\zeta(z) = \frac{4\delta}{-3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}} \frac{\mathrm{dn}^2(\mu z, m) + m - 1}{m\,\mathrm{dn}^2(\mu z, m)},$$

$$\mu = \frac{1}{2}\sqrt{-3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}},$$

$$m = \frac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{3\gamma - \sqrt{9\gamma^2 - 16\beta\delta}}. \tag{20}$$

Solutions of this type are not considered in [1].

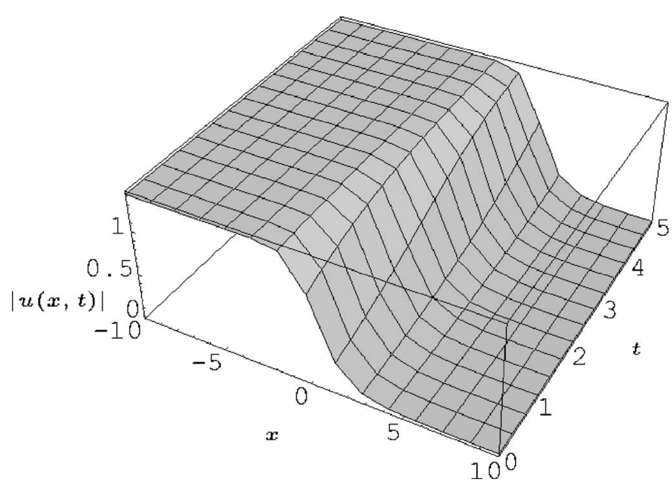


FIG. 1. Kink solitary wave like solution [cf. Eq. (23)] for $\alpha = 1$, $\beta = -1$, and $v = 1/2$.

Solving $\Delta=0$ subject to $\alpha>0$, $\beta<0$, and the PDC, yields various possibilities. One is given by $\gamma=2\beta^2/3\alpha$, $\delta=\epsilon=0$ ($E>0$, $D<0$, $C=D^2/4E$, $K_1=K_2=0$ in Ref. [1]). According to Eqs. (4)–(6) and conditions (8) a solitary solution is given by this choice of parameters. Because Eq. (1) has two double roots $\zeta_{1,2}=0$, $\zeta_{3,4}=-2\beta/\alpha$, Eq. (2) instead of Eq. (3) has to be used to evaluate this solitary solution. Inserting $\zeta_0=-\beta/\alpha$ into Eq. (2) and using the relation [8]

$$\wp(z)=e_1+\frac{3e_1}{\sinh^2(\sqrt{3e_1}z)},$$

$$g_2>0,\quad g_3<0,\quad \Delta=0, \tag{21}$$

leads to the kink solitary solution

$$\zeta(z)=-\frac{\beta}{\alpha}\left[1+\tanh\left(\frac{\beta}{\sqrt{\alpha}}z\right)\right], \tag{22}$$

The setting $u(x,t)=a(x,t)\exp[i\varphi(x,t)]$, $a^2=\zeta$, $z=x-vt$ in Ref. [1] yields

$$|u(x,t)|=\sqrt{-\frac{\beta}{\alpha}\left\{1+\tanh\left[\frac{\beta}{\sqrt{\alpha}}(x-vt)\right]\right\}}. \tag{23}$$

This solution is shown in Fig. 1 for $\alpha=1$, $\beta=-1$, and $v=1/2$. It should be mentioned that a kink solitary wave solution $\zeta$ is not given in [1].

It is the purpose of this comment to point out that all solutions presented in [1] are particular cases of the general solution (2) that can be evaluated simply by the PDC. Subject to ansatz (KL.3) Eq. (2) represents all elliptic solutions of the DNSE, since Eq. (1) is solved uniquely by $\zeta(z)$ according to Eq. (2). In conclusion, the trial function approach to solving Eq. (KL.9) seems to be *ad hoc* and does not yield the complete set of elliptic solutions of the DNSE.

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